p -QUASİ-KONVEKS FONKSİYONLAR İÇİN GENELLEŞTİRİLMİŞ HERMİTE-HADAMARD TİPLİ EŞİTSİZLİKLER

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Özet

Bu çalışmada, yazar türevlenebilir fonksiyonlar için yeni genel bir özdeşlik verir ve bu özdeşliği kullanarak p-quasi konveks fonksiyonlar için bazı yeni genelleştirilmiş Hermite-Hadamard tipli eşitsizlikler elde eder.

Mathematics Subject Classification: 26D15, 26A51

Anahtar Kelimeler: Hermite-Hadamard eşitsizliği, p -quasi konveks fonksiyon

Generalized Hermite-Hadamard Type Inequalities for p -Quasi-Convex Functions

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Abstract

In this paper, the author gives a new general identity for differentiable functions and establishes some new generalized Hermite-Hadamard type inequalities for p -quasi convex functions by using this identity.

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1 Introduction

Let \( f : I \subset \mathbb{R} \to \mathbb{R} \) be a convex function defined on the interval \( I \) of real numbers and \( a, b \in I \) with \( a < b \). The following inequality

\[
f\left(\frac{a+b}{2}\right) \leq \frac{1}{b-a} \int_a^b f(x)dx \leq \frac{f(a) + f(b)}{2}
\]

holds. This double inequality is known in the literature as Hermite-Hadamard integral inequality for convex functions. Note that some of the classical inequalities for means can be derived from (1) for appropriate particular selections of the mapping \( f \). Both inequalities hold in the reversed direction if \( f \) is concave.

In Dragomir & Agarwal (1998), gave the following Lemma. By using this Lemma, Dragomir obtained the following Hermite-Hadamard type inequalities for convex functions:

**Lemma 1** Let \( f : I^* \subset \mathbb{R} \to \mathbb{R} \) be a differentiable mapping on \( I^* \) and \( a, b \in I^* \) with \( a < b \). If \( f' \in L[a,b] \), then the following equality holds:

\[
\frac{f(a) + f(b)}{2} - \frac{1}{b-a} \int_a^b f(x)dx = \frac{b-a}{2} \int_0^1 (1-2t)f'(ta + (1-t)b)dt.
\]

The notion of quasi-convex functions generalizes the notion of convex functions. More precisely, a function \( f : [a,b] \to \mathbb{R} \) is said quasi-convex on \( [a,b] \) if

\[
f(\alpha x + (1-\alpha)y) \leq \sup \{f(x), f(y)\},
\]

for any \( x, y \in [a,b] \) and \( \alpha \in [0,1] \). Clearly, any convex function is a quasi-convex function. Furthermore, there exist quasi-convex functions which are not convex (see Ion 2007).

For some results which generalize, improve and extend the inequalities(1) related to quasi-convex functions we refer the reader to see (Alomari et al 2010; Alomari et al 2011; Ion 2007; İşcan 2013; 2013; 2013, İşcan et al 2014,Zehang 2013) and plenty of references therein.

In (İşcan 2014), the author, gave definition Harmonically convex and concave functions as follow.

**Definition 1** Let \( I \subset \mathbb{R} \setminus \{0\} \) be a real interval. A function \( f : I \to \mathbb{R} \) is said to be harmonically convex, if

\[
f\left(\frac{xy}{tx + (1-t)y}\right) \leq tf(y) + (1-t)f(x)
\]

(3)
for all \( x, y \in I \) and \( t \in [0,1] \). If the inequality in (3) is reversed, then \( f \) is said to be harmonically concave.

Zhang et al (2013) defined the harmonically quasi-convex function and supplied several properties of this kind of functions.

**Definition 2** A function \( f : I \subseteq (0, \infty) \rightarrow [0, \infty) \) is said to be harmonically quasi-convex, if

\[
\frac{xy}{tx+(1-t)y} \leq \sup \{f(x), f(y)\}
\]

for all \( x, y \in I \) and \( t \in [0,1] \).

We would like to point out that any harmonically convex function on \( I \subseteq (0, \infty) \) is a harmonically quasi-convex function, but not conversely. For example, the function

\[
f(x) = \begin{cases} 
1, & x \in (0,1]; \\
(x-2)^2, & x \in [1,4]. 
\end{cases}
\]

is harmonically quasi-convex on \((0,4]\), but it is not harmonically convex on \((0,4]\). In [16], Zhang and Wan gave definition of \( p \)-convex function as follow:

**Definition 3** Let \( I \) be a \( p \)-convex set. A function \( f : I \rightarrow R \) is said to be a \( p \)-convex function or belongs to the class \( PC(I) \), if

\[
f\left(\left[\alpha x^p + (1-\alpha) y^p \right]^{1/p}\right) \leq \alpha f(x) + (1-\alpha) f(y)
\]

for all \( x, y \in I \) and \( \alpha \in [0,1] \).

**Remark 1** An interval \( I \) is said to be a \( p \)-convex set if \( \left[\alpha x^p + (1-\alpha) y^p \right]^{1/p} \in I \) for all \( x, y \in I \) and \( \alpha \in [0,1] \), where \( p = 2k + 1 \) or \( p = n/m \), \( n = 2r + 1 \), \( m = 2t + 1 \) and \( k, r, t \in \mathbb{N} \).

**Remark 2** If \( I \subseteq (0, \infty) \) be a real interval and \( p \in R \setminus \{0\} \), then

\[
\left[\alpha x^p + (1-\alpha) y^p \right]^{1/p} \in I \text{ for all } x, y \in I \text{ and } \alpha \in (0,1].
\]

According to Remark 2, we can give a different version of the definition of \( p \)-convex function as follow:
**Definition 4** [10,11,12] Let $I \subset (0, \infty)$ be a real interval and $p \in R \setminus \{0\}$. A function $f : I \to \mathbb{R}$ is said to be a p-convex function, if

$$f \left[ \alpha x^p + (1 - \alpha) y^p \right]^{1/p} \leq \alpha f(x) + (1 - \alpha) f(y)$$

(4)

for all $x, y \in I$ and $\alpha \in [0,1]$. If the inequality in (4) is reversed, then $f$ is said to be $p$-concave.

According to Definition 4, it can be easily seen that for $p = 1$ and $p = -1$, $p$-convexity reduces to ordinary convexity and harmonically convexity of functions defined on $I \subset (0, \infty)$, respectively.

**Example 1**

Let $f : (0, \infty) \to R$, $f(x) = x^p$, $p \neq 0$, and $g : (0, \infty) \to R$, $g(x) = c$, $c \in R$, then $f$ and $g$ are both $p$-convex and $p$-concave functions.

In [4, Theorem 5], if we take $I \subset (0, \infty)$, $p \in R \setminus \{0\}$ and $h(t) = t$, then we have the following Theorem.

**Theorem 1**

Let $f : I \subset (0, \infty) \to R$ be a $p$-convex function, $p \in R \setminus \{0\}$, and $a, b \in I$ with $a < b$. If $f \in L[a,b]$ then we have

$$f \left( \left[ a^p + b^p \right]^{\frac{1}{p}} \right) \leq \frac{p}{b^p - a^p} \int_a^b f(x) \frac{1}{x^{1+p}} \, dx \leq \frac{f(a) + f(b)}{2}.$$ 

(5)

In [11], İşcan defined the $p$-quasi-convex function and supplied several properties of this kind of functions as follow:

**Definition 5**

Let $I \subset (0, \infty)$ be a real interval and $p \in R \setminus \{0\}$. A function $f : I \to R$ is said to be $p$-quasi-convex, if

$$f \left[ t x^p + (1-t) y^p \right]^{1/p} \leq \max \{f(x), f(y)\}$$

(6)

for all $x, y \in I$ and $t \in [0,1]$. If the inequality in (6) is reversed, then $f$ is said to be $p$-quasi-concave.

It can be easily seen that for $r = 1$ and $r = -1$, $p$-quasi convexity reduces to ordinary quasi convexity and harmonically quasi convexity of functions defined on $I \subset (0, \infty)$, respectively. Moreover every $p$-convex function is a $p$-quasi-convex function.
Example 2 Let $f : (0, \infty) \to R, f(x) = x^p, p \in R \backslash \{0\}$, and $g : (0, \infty) \to R, g(x) = c, c \in R$, then $f$ and $g$ are $p$-quasi-convex functions.

Proposition 1 Let $I \subset (0, \infty)$ be a real interval, $p \in R \backslash \{0\}$ and $f : I \to R$ is a function, then:

1. If $p \leq 1$ and $f$ is quasi-convex and nondecreasing function then $f$ is $p$-quasi-convex.
2. If $p \geq 1$ and $f$ is $p$-quasi-convex and nondecreasing function then $f$ is quasi-convex.
3. If $p \leq 1$ and $f$ is $p$-quasi-concave and nondecreasing function then $f$ is quasi-concave.
4. If $p \geq 1$ and $f$ is quasi-concave and nondecreasing function then $f$ is $p$-quasi-concave.
5. If $p \geq 1$ and $f$ is quasi-convex and nonincreasing function then $f$ is $p$-quasi-convex.
6. If $p \leq 1$ and $f$ is $p$-quasi-convex and nonincreasing function then $f$ is quasi-convex.
7. If $p \geq 1$ and $f$ is $p$-quasi-concave and nonincreasing function then $f$ is quasi-concave.
8. If $p \leq 1$ and $f$ is quasi-concave and nonincreasing function then $f$ is $p$-quasi-concave.

Proposition 2 If $f : [a,b] \subseteq (0, \infty) \to R$ and if we consider the function $g : [a^p,b^p] \to R$, defined by $g(t) = f(t^{1/p})$, $p \neq 0$, then $f$ is $p$-quasi-convex on $[a,b]$ if and only if $g$ is quasi-convex on $[a^p,b^p]$.

For some results related to $p$-convex functions and its generalizations, we refer the reader to see (Fang 2014; İşcan 2016; 2016; 2016, Noor 2015; Zhang et al 2015).

The main purpose of this paper is to establish some new general results connected with the right-hand side of the inequalities (5) for $p$-quasi-convex functions.

2 Main Results

In order to prove our main results we need the following lemma:
Lemma 2 Let \( f : I \subset (0, \infty) \to R \) be a differentiable mapping on \( I^\circ, \ a, b \in I \) with \( a < b \). If \( f' \in L[a, b], \ p \in R \setminus \{0\} \) and \( \lambda, \mu \in [0, \infty), \ \lambda + \mu > 0 \), then the following equality holds:

\[
\frac{\lambda f(a) + \mu f(b)}{\lambda + \mu} - \frac{p}{b^p - a^p} \int_a^b f(x) \frac{b}{x^{1-p}} dx = \frac{b^p - a^p}{p(\lambda + \mu)} \int_0^1 \left[ (\lambda + \mu)t - \lambda \right] f'(M_{p, \lambda}(a, b)) dt
\]

where \( M_{p, \lambda}(a, b) = \left[ b^p + (1-t)a^p \right]^{1/p} \).

Proof: integration by parts we have

\[
I = \frac{b^p - a^p}{p(\lambda + \mu)} \int_0^1 \left[ (\lambda + \mu)t - \lambda \right] f'(M_{p, \lambda}(a, b)) dt
\]

\[
= \frac{1}{(\lambda + \mu)} \left[ (\lambda + \mu)t - \lambda \right] df(M_{p, \lambda}(a, b))
\]

\[
= \frac{(\lambda + \mu)t - \lambda}{(\lambda + \mu)} f(M_{p, \lambda}(a, b)) - \int_0^1 f(M_{p, \lambda}(a, b)) dt
\]

Setting \( x^p = tb^p + (1-t)a^p \), and \( px^{p-1} dt = (b^p - a^p) dt \) gives

\[
I = \frac{\lambda f(a) + \mu f(b)}{\lambda + \mu} - \frac{p}{b^p - a^p} \int_a^b f(x) \frac{b}{x^{1-p}} dx.
\]

which completes the proof.

Remark 3 If we take \( \lambda = \mu = p = 1 \) in Lemma 2, then we obtain the inequality (2) in Lemma 1.

Theorem 2 Let \( f : I \subset (0, \infty) \to R \) be a differentiable function on \( I^\circ, \ a, b \in I^\circ \) with \( a < b \), \( p \in R \setminus \{0\} \) and \( f' \in L[a, b] \). \( f \mid f' \) is \( p \)-convex on \( [a, b] \), then

\[
\left| \frac{\lambda f(a) + \mu f(b)}{\lambda + \mu} - \frac{p}{b^p - a^p} \int_a^b f(x) \frac{b}{x^{1-p}} dx \right|
\]
\[ \frac{b^p - a^p}{p(\lambda + \mu)} \left( \max \{ f'(a), |f'(b)| \} \right) C_{\lambda, \mu}(a, b; p) \]

where

\[ C_{\lambda, \mu}(a, b; p) = \frac{2p(\lambda + \mu)}{(p+1)(b^p - a^p)^2} \left[ \frac{A}{\lambda + \mu} \left( M_{\lambda, \mu}^p - A \right) \right. \]

\[ \left. - (p+1) \left( M_{\lambda, \mu}^{p+1} - A_{1}^{p+1} \right) \right], \quad p \in P \setminus \{-1, 0\}, \]

and

\[ M_{p,q}(a, b) = \left[ t b^p + (1-t)a^p \right]^{\frac{1}{l-p}}, \quad A_{0,1} = t b^p + (1-t)a^p, \quad M_{t,0} = A_{t,0}^p, \quad A = (a+b)/2 \quad \text{and} \quad G = \sqrt{ab}. \]

**Proof:** From Lemma 2 and using the Hölder integral inequality and \( p \)-quasi-convexity of \( |f'| \) on \([a, b]\), we have

\[ \left| \frac{\lambda f(a) + \mu f(b)}{\lambda + \mu} - \frac{p}{b^p - a^p} \int_{a}^{b} f(x) x^\lambda dx \right| \]

\[ \leq \frac{b^p - a^p}{p(\lambda + \mu)} \left( \int_{0}^{1} \left| \frac{\lambda + \mu t - \lambda}{t b^p + (1-t)a^p} \right|^\frac{1}{l-p} dt \right) \left( \max \{ f'(a), |f'(b)| \} \right) \]

\[ \int_{0}^{1} \left| \frac{\lambda + \mu t - \lambda}{t b^p + (1-t)a^p} \right|^\frac{1}{l-p} dt = C_{\lambda, \mu}(a, b; p). \]
In Theorem 2, if we put $p = 1$, then we obtain the following corollary for quasi-convex functions:

**Corollary 1** Under the conditions of Theorem 2, if we take $p = 1$, then we have

$$
\left| \frac{\lambda f(a) + \mu f(b)}{\lambda + \mu} - \frac{1}{b-a} \int_{a}^{b} f(x) dx \right| 
\leq \frac{b-a}{\lambda + \mu} \left( \max \{|f'(a)|, |f'(b)|\} \right) C_{\lambda,\mu}(a, b; 1).
$$

In Theorem 2, if we put $p = -1$, then we obtain the following corollary for harmonically quasi-convex functions:

**Corollary 2** Under the conditions of Theorem 2, if we take $p = -1$, then we have

$$
\left| \frac{\lambda f(a) + \mu f(b)}{\lambda + \mu} - \frac{ab}{b-a} \int_{a}^{b} f(x) dx \right| 
\leq \frac{b-a}{\lambda + \mu} \left( \max \{|f'(a)|, |f'(b)|\} \right) C_{\lambda,\mu}(a, b; -1).
$$

**Theorem 3** Let $f : I \subset (0, \infty) \to R$ be a differentiable function on $I^*$, $a, b \in I^*$ with $a < b$, $p \in R \setminus \{0\}$ and $f' \in L[a, b]$. If $|f'|^q$ is $p$-convex on $[a, b]$ for $q > 1$, $\frac{1}{r} + \frac{1}{q} = 1$, then

$$
\left| \frac{\lambda f(a) + \mu f(b)}{\lambda + \mu} - \frac{p}{b^p - a^p} \int_{a}^{b} f(x) dx \right| 
\leq \frac{b^p - a^p}{2p} K_{\lambda,\mu}^{1/r}(r)D^{1/q}(a, b; p; q) \left( \max \{|f'(a)|^q, |f'(b)|^q\} \right)^{1/q}
$$

where

$$
K_{\lambda,\mu}(r) = \frac{\lambda^{r+1} + \mu^{r+1}}{(r+1)(\lambda + \mu)},
$$

$$
D(a, b; p; q) = \begin{cases} 
L_{p}^{-1}L_{q-p+1}^{-1}, & p \in R \setminus \{0, 1, q \} \cup \{q-1\} \\
L^1(a^p, b^p), & p = q \cup \{q-1\} \\
1, & p = 1
\end{cases}
$$
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\[ L_p = L_p(a,b) := \left( \frac{b^{p+1} - a^{p+1}}{(p+1)(b-a)} \right)^{\frac{1}{p}}, \quad p \in \mathbb{R} \setminus \{-1,0\}, \]

\[ L(a,b) := \frac{b-a}{\ln b - \ln a} \]

is the \( p \)-logarithmic mean and \( \log \) mean.

**Proof:** From Lemma 2 and using the Hölder integral inequality and \( p \)-quasi-convexity of \( |f|^q \) on \([a,b]\), we have

\[
\frac{\lambda f(a) + \mu f(b)}{\lambda + \mu} - \frac{p}{b^p - a^p} \int_a^b f(x) \left( x^{1-p} \right) dx \\
\leq \frac{b^p - a^p}{p(\lambda + \mu)} \left[ \int_0^1 \left( \frac{\lambda + \mu}{tb^p + (1-t)a^p} \right)^{1-\frac{1}{q}} \left( M_{p,r}(a,b) \right)^q dt \right]^{\frac{1}{q}} \\
\leq \frac{b^p - a^p}{2p} \left[ \int_0^1 \left( \frac{\lambda + \mu}{tb^p + (1-t)a^p} \right)^{1-\frac{1}{q}} \left( M_{p,r}(a,b) \right)^q dt \right]^{\frac{1}{q}} \\
\leq \frac{b^p - a^p}{2p} K_{\lambda,\mu}^{\frac{1}{r},q} \left( \max \left\{ f'(a)^q, f'(b)^q \right\} \right)^{\frac{1}{q}} \\
\leq \frac{b^p - a^p}{2p} K_{\lambda,\mu}^{\frac{1}{r},q} D_{a,b}^{\frac{1}{r},q} \left( \max \left\{ f'(a)^q, f'(b)^q \right\} \right)^{\frac{1}{q}}.
\]

It is easily check that

\[
\int_0^1 \left( \frac{\lambda + \mu}{tb^p + (1-t)a^p} \right) dt = K_{\lambda,\mu}(r),
\]

\[
\int_0^1 \left( \frac{1}{tb^p + (1-t)a^p} \right)^{\frac{1}{q}} dt = D(a,b; p; q).
\]

In Theorem 3, if we put \( p = 1 \), then we obtain the following corollary for quasi-convex functions:

**Corollary 3** Under the conditions of Theorem 2, if we take \( p = 1 \), then we have
\[
\left| \frac{\lambda f(a) + \mu f(b)}{\lambda + \mu} - \frac{ab}{b-a} \int_a^b \frac{f(x)}{x^q} \, dx \right| \\
\leq \frac{b-a}{2} K^{l/r}_{\lambda, \mu}(r) \left( \max \left\{ \left| f'(a) \right|^q, \left| f'(b) \right|^q \right\} \right)^{1/q}.
\]

In Theorem 3, if we put \( p = -1 \), then we obtain the following corollary for harmonically quasi-convex functions:

**Corollary 4** Under the conditions of Theorem 2, if we take \( p = -1 \), then we have

\[
\left| \frac{\lambda f(a) + \mu f(b)}{\lambda + \mu} - \frac{ab}{b-a} \int_a^b \frac{f(x)}{x^q} \, dx \right| \\
\leq \frac{b^p - a^p}{2p} K^{l/r}_{\lambda, \mu}(r) D^{l/q}(a, b; -1; q) \left( \max \left\{ \left| f'(a) \right|^q, \left| f'(b) \right|^q \right\} \right)^{1/q}.
\]

**References**


